OPERATIONS ON POLYHEDRAL PRODUCTS AND A NEW TOPOLOGICAL CONSTRUCTION OF INFINITE FAMILIES OF TORIC MANIFOLDS

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Abstract. A combinatorial construction is used to analyze the properties of polyhedral products and generalized moment-angle complexes with respect to certain operations on CW pairs including exponentiation. This allows for the construction of infinite families of toric manifolds in a way which simplifies the combinatorial input and consequently, the presentation of the cohomology rings. A noteworthy example, described in Section 5, recovers the cohomology ring of *every* complex projective space from the information in a one-dimensional fan. Various applications of the ideas introduced here have appeared already in the literature.

Contents

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1. Introduction

The polyhedral product $Z(K; (\underline{X}, \underline{A}))$ is a CW-complex valued functor of two variables: the first, an abstract simplicial complex K on m vertices and the second, a family of (based) CW pairs

$$(\underline{X}, \underline{A}) \ = \ \{(X_1, A_1), (X_2, A_2), \ldots, (X_m, A_m)\}.$$

It is defined as a union of products inside $\prod\limits_{i=1}^{m} X_i$ each parameterized by a simplex in K.

Polyhedral products generalize the spaces called *moment-angle complexes* which were developed first by Buchstaber and Panov in [5] and correspond to the case $(X_i, A_i) = (D^2, S^1)$, $i = 1, 2, \ldots, m$.

A part of this paper is devoted to an analysis of the properties of the polyhedral product functor with respect to an operation which associates to a simplicial complex K and a sequence of positive integers $J = (j_1, j_2, \ldots, j_m)$, a new and larger simplicial complex $K(J)$. A key result, Theorem 7.2, relates this construction to the exponentiation operation on CW pairs. This result has unexpected consequences for the case

$$(\underline{X}, \underline{A}) \ = \ (\underline{D}^{2J}, \underline{S}^{2J-1}) \ = \ \big\{(D^{2j_i}, S^{2j_i-1})\big\}_{i=1}^{m}$$

corresponding to the polyhedral products which are called now *generalized* moment-angle complexes. The application links these spaces to the study of toric manifolds in a novel way. This is illustrated by the construction of *every* projective space from the fan information for the first one $\mathbb{C}P^1$.

For the much studied case $(X_i, A_i) = (D^1, S^0)$, $i = 1, 2, \ldots, m$, there is the surprising observation in Section 7 that every moment-angle complex $Z(K; (D^2, S^1))$ can be realized as $Z\big(K(J); (D^1, S^0)\big)$ for $J = (2, 2, \ldots, 2)$. So, in a certain sense, the "real" moment-angle complex is the more basic object.

In the context of toric manifolds over a simple polytope P^n, with dual bounding simplicial complex $K = \partial P^n$, the spaces $Z(\partial P^n; (D^2, S^1))$ and $Z(\partial P^n; (D^1, S^0))$ were introduced by Davis and Januszkiewicz in [7]. They used the latter to construct spaces known as *small covers*, the subject of considerable current investigation.

To describe the ideas further, recall that a toric manifold M^{2n} is a manifold covered by local charts $\mathbb{C}^n$, each with the standard action of a real n-dimensional torus T^n, compatible in such a way that the quotient $M^{2n}\big/T^n$ has the structure of a *simple* polytope P^n. Under the T^n action, each copy of $\mathbb{C}^n$ must project to an $\mathbb{R}^n_+$ neighborhood of a vertex of P^n.

The fundamental construction of Davis and Januszkiewicz [7, Section 1.5], realizes all toric manifolds and in particular, all smooth projective toric varieties. From this construction it

follows that M^{2n} can be realized as the quotient of an $(m+n)$-dimensional moment-angle complex by the free action of a certain real $(m-n)$-dimensional torus $T^{m-n} \subset T^m$. This subtorus is specified usually by a *characteristic map* λ.

Beginning with a toric manifold M^{2n}, its associated m faceted simple polytope P^n and characteristic map λ, an infinite family of new toric manifolds $M(J)$ is constructed, one for each sequence of positive integers $J = (j_1, j_2, \ldots, j_m)$. The manifolds $M(J)$ are determined by a new polytope $P(J)$ and a new characteristic map $\lambda(J)$. The unexpected outcome here (Theorem 4.2), is that the integral cohomology ring of $M(J)$ is described completely in terms of the original map λ and the original polytope P^n. The manifolds $M(J)$ are a rich new class of toric manifolds which come equipped with a complete fan, (where appropriate), combinatorial and topological information. These spaces are a new, systematic infinite family of toric manifolds which have tractable as well as natural properties. .

In Section 9, the construction and properties of these manifolds $M(J)$ are analyzed in the context of *generalized* moment-angle complexes. As above, to the polytope P^n is associated its dual complex K and a generalized moment-angle complex $Z(K;(\underline{D}^{2J},\underline{S}^{2J-1}))$ and, to $P(J)$, which happens to have dual complex $K(J)$, is associated the moment-angle complex $Z\big(K(J);(D^2,S^1)\big)$. The central result which here connects the construction $K(J)$ to the study of toric manifolds is the following.

Theorem 7.5. *There is an action of T^m on both $Z(K;(\underline{D}^{2J},\underline{S}^{2J-1}))$ and $Z\big(K(J);(D^2,S^1)\big)$, with respect to which they are equivariantly diffeomorphic.*

In describing the new manifolds $M(J)$, diffeomorphisms, (Theorems 7.5 and 9.2),

$$Z\big(K(J);(D^2,S^1)\big)/T^{m-n} \longrightarrow M(J) \longleftarrow Z(K;(\underline{D}^{2J},\underline{S}^{2J-1}))/T^{m-n},$$

mirror geometrically the reduction in combinatorial complexity from the pair $\big(\lambda(J),P(J)\big)$ to the pair (λ, P^{2n}). Significant in the theory of toric manifolds is the role played by the Davis-Januszkiewicz spaces. These are homotopy equivalent to polyhedral products of the form $Z(K;(\mathbb{C}P^\infty,*))$. Key in the theory which is developed here, are related spaces, denoted by $Z\big(K;(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{J-1})\big)$. They substitute for the usual Davis-Januszkiewicz spaces $Z\big(K(J);(\mathbb{C}P^\infty,*)\big)$. Their properties and relationship to the manifolds $M(J)$ are discussed in Theorems 10.5 and 11.1. In contrast to the cohomology of the Davis-Januszkiewicz spaces, these spaces have integral cohomology rings which are monomial ideal rings but the monomials are not necessarily square-free.

The construction of the toric manifolds $M(J)$ leads to the idea of *nests* which is discussed in the final section. The most obvious example is given by the projective spaces

$$\mathbb{C}P^1 \subset \mathbb{C}P^2 \subset \cdots \subset \mathbb{C}P^k \subset \cdots$$

a nest of codimension-two embeddings each with normal bundle the canonical line bundle over $\mathbb{C}P^k$. An ordering exists on on sequences $J = (j_1, j_2 \ldots, j_m)$ and $L = (l_1, l_2, \ldots, l_m)$ so that if $J < L$ there is a natural embedding $M(J) \longrightarrow M(L)$. Moreover, the normal bundle of the embedding is a sum of canonical complex line bundles determined by the two sequences L and J.

Remark. Unless indicated otherwise, all cohomology rings throughout are considered to be with integral coefficients.

The intersections of certain quadrics are known to be diffeomorphic to moment-angle manifolds, [4], [11] and implicitly in [5, Construction 3.1.8]. After a first draft of this article was written in 2008, the authors learned of the work of S. Lopez de Medrano on the intersections of quadrics, [14], [13]. Results in [13] depend on the consequences of a doubling of variables and a duplication of coefficients in the defining equations; this translates into an instance of our general construction.

Following lectures by the authors on the material contained in this paper, several applications of the ideas have appeared both in the literature and in preprint form. These include [16], [17], [10] and [12].

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2. The simplicial wedge construction

Let K be a simplicial complex of dimension $n-1$ on vertices $\{v_1, v_2, \ldots, v_m\}$ and let $J = (j_1, j_2, \ldots, j_m)$ be a sequence of positive integers.

Definition 2.1. A *non-face* of a simplicial complex K is a simplex τ whose vertices are in K but the simplex itself is not. It is *minimal* if K contains any proper face of τ. Let K be as above. Denote by $K(J)$ the simplicial complex on vertices

$$\{\underbrace{v_{11}, v_{12}, \ldots, v_{1j_1}}, \underbrace{v_{21}, v_{22}, \ldots, v_{2j_2}}, \ldots, \underbrace{v_{m1}, v_{m2}, \ldots, v_{mj_m}}\}$$

with the property that

$$\{\underbrace{v_{i_11}, v_{i_12}, \ldots, v_{i_1j_{i_1}}}, \underbrace{v_{i_21}, v_{i_22}, \ldots, v_{i_2j_{i_2}}}, \ldots, \underbrace{v_{i_k1}, v_{i_k2}, \ldots, v_{i_kj_{i_k}}}\}$$

is a minimal non-face of $K(J)$ if and only if $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ is a minimal non-face of K. Moreover, all minimal non-faces of $K(J)$ have this form.

An alternative construction of the simplicial complex $K(J)$ will reveal the fact that $K(J)$ is the boundary complex of a simple polytope $P(J)$ if K is the boundary complex of a simple polytope P^n. Recall that for $\sigma \in K$, the *link* of σ in k, is the set

$$\operatorname{link}_K\sigma := \{\tau \in K \colon \sigma \cup \tau \in K, \sigma \cap \tau = \emptyset\}.$$

The *join* of two simplicial complexes K_1, K_2 on disjoint vertex sets S_1 and S_2 respectively is given by

$$K_1 * K_2 := \{\sigma \subset S_1 \cup S_2 \colon \sigma = \sigma_1 \cup \sigma_2,\ \sigma_1 \in K_1,\ \sigma_2 \in K_2\}.$$

Construction 2.2. As above, let K be the simplicial complex on vertices $\{v_1, v_2, \ldots, v_m\}$. Choose a fixed vertex v_i in K and consider $\Delta^1 = \big\{\{v_{i1}\}, \{v_{i2}\}, \{v_{i1}, v_{i2}\}\big\}$, the simplicial complex which is a one-simplex. Define a new simplicial complex $K(v_i)$ on $m+1$ vertices by

$$K(v_i) := \{v_{i1}, v_{i2}\} * \operatorname{link}_K\{v_i\} \ \cup\ \big\{\{v_{i1}\}, \{v_{i2}\}\big\} * (K \setminus \{v_i\}). \tag{2.1}$$

The vertex v_i does not appear in the vertex set of $K(v_i)$. The vertices of $K(v_i)$, other than v_{i1} and v_{i2}, are re-labelled by setting $v_k = v_{k1}$ if $k \neq i$. So, the new vertex set of $K(v_i)$ has become

$$S = \{v_{11}, \ldots, v_{(i-1)1}, v_{i1}, v_{i2}, v_{(i+1)1}, \ldots, v_{m1}\}.$$

Example 2.3. The easiest example is that of $K = \{v_1\}, \{v_2\}$ two disjoint points. Here, $K(v_1)$ has three vertices $\{v_{11}, v_{11}, v_{21}\}$, $\text{link}_K\{v_1\} = \varnothing$ and $K \setminus \{v_1\} = v_2$. So (2.1) becomes

$$\{v_{11}, v_{12}\} * \varnothing \,\cup\, \big\{\{v_{i1}\}, \{v_{i2}\}\big\} * \{v_2\} \;=\; \{v_{11}, v_{12}\} \,\cup\, \big(\{v_{11}, v_{21}\} \,\cup\, \{v_{11}, v_{21}\}\big) \tag{2.2}$$

which is the boundary of a two-simplex.

In the paper [15, page 578], this construction is called the *simplicial wedge* of K on v. Notice that if $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ is a minimal non-simplex of K with $i_j \neq i$ for all j, then it remains a minimal non-simplex of $K(v_i)$. The simplex $\{v_{i1}, v_{i2}\}$ becomes part of a simplex

$$\{v_{i_1 1}, v_{i_2 1}, \ldots, v_{i_k 1}, v_{i1}, v_{i2}, v_{i_{(k+1)}1}, \ldots, v_{i_s 1}\} \;\in\; \{v_{i1}, v_{i2}\} * \text{link}_K\{v_i\} \;\subseteq\; K(v_i)$$

if and only if

$$\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}, v_i, v_{i_{k+1}}, \ldots, v_{i_s}\} \in K.$$

Hence, according to Definition 2.1, $K(v_i) = K(J)$, where

$$J = (1, 1, \ldots, 1, \overset{i}{2}, 1, \ldots, 1),$$

is the m-tuple with 2 appearing in the i^{th} spot. According to [15, page 582], $K(v_i)$ is dual to a simple polytope $P(v_i)$ of dimension $n+1$ with $m+1$ facets if K is dual to a simple polytope P^n of dimension n with m facets. Beginning with $J = (1, 1, \ldots, 1)$, Construction 2.2 may be iterated to produce $K(J)$ for any $J = (j_1, j_2, \ldots, j_m)$. The induction from $J = (j_1, j_2, \ldots, j_m)$ to $J' = (j_1, j_2, \ldots, j_{i-1}, j_i + 1, j_{i+1}, \ldots, j_m)$, necessitates a choice of vertex v from among $\{v_{i1}, v_{i2}, \ldots, v_{ij_i}\}$ in order to form $K(J)(v)$, as in Construction 2.2. The fundamental property, described in Definition 2.1, ensures that any choice of v, will result in precisely the same minimal non-simplices in $K(J)(v) = K(J')$.

The next theorem follows from these observations. Set $d(J) = j_1 + j_2 + \cdots + j_m$.

Theorem 2.4. *Let $J = (j_1, j_2, \ldots, j_m)$ and suppose K is dual to a simple polytope P^n having m facets. Then $K(J)$ is dual to a simple polytope $P(J)$ of dimension $d(J) - m + n$ having $d(J)$ facets.* □

This section ends with a simple necessary criterion for a simplicial complex to be in the image of the simplicial wedge construction. The condition follows immediately from the definition of the construction

Remark. If a simplicial complex K' exists satisfying, $K = K'(v)$ for some $\{v\} \in K'$, then K must contain vertices v_1 and v_2 satisfying:

(1) the one-simplex $\{v_1, v_2\} \in K$ and
(2) interchanging v_1 and v_2 is a simplicial automorphism of K.

3. New toric manifolds made from a given one

As described in the Introduction, a toric manifold M^{2n} is a manifold covered by local charts $\mathbb{C}^n$, each with the standard action of a real n-dimensional torus T^n, compatible in such a way that the quotient $M^{2n}\big/T^n$ has the structure of a *simple* polytope P^n. Here, "simple" means that P^n has the property that at each vertex, exactly n facets intersect. Under the T^n action, each copy of $\mathbb{C}^n$ must project to an $\mathbb{R}^n_+$ neighborhood of a vertex of P^n. The fundamental construction of Davis and Januszkiewicz [7, Section 1.5] is described briefly below. It realizes all toric manifolds and, in particular, all smooth projective toric varieties. Let

$$\mathcal{F} = \{F_1, F_2, \dots, F_m\}$$

denote the set of facets of P^n. The fact that P^n is simple implies that every codimension-l face F can be written uniquely as

$$F = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l}$$

where the F_{i_j} are the facets containing F. Let

$$\lambda : \mathcal{F} \longrightarrow \mathbb{Z}^n \tag{3.1}$$

be a function into an n-dimensional integer lattice satisfying the condition that whenever $F = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l}$ then $\{\lambda(F_{i_1}), \lambda(F_{i_2}), \dots, \lambda(F_{i_l})\}$ span an l-dimensional submodule of $\mathbb{Z}^n$ which is a direct summand. Such a map is called a *characteristic function* associated to P^n. Next, regarding $\mathbb{R}^n$ as the Lie algebra of T^n, the map λ is used to associate to each codimension-l face F of P^n a rank-l subgroup $G_F \subset T^n$. Specifically, writing $\lambda(F_{i_j}) = (\lambda_{1i_j}, \lambda_{2i_j}, \dots, \lambda_{ni_j})$ gives

$$G_F = \big\{\big(e^{2\pi i(\lambda_{1i_1}t_1 + \lambda_{1i_2}t_2 + \cdots + \lambda_{1i_l}t_l)}, \dots, e^{2\pi i(\lambda_{ni_1}t_1 + \lambda_{ni_2}t_2 + \cdots + \lambda_{ni_l}t_l)}\big) \in T^n\big\}$$

where $t_i \in \mathbb{R}$, $i = 1, 2, \ldots, l$. Finally, let $p \in P^n$ and $F(p)$ be the unique face with p in its relative interior. Define an equivalence relation $\sim$ on $T^n \times P^n$ by $(g, p) \sim (h, q)$ if and only if $p = q$ and $g^{-1}h \in G_{F(p)} \cong T^l$. Then

$$M^{2n} \cong M^{2n}(\lambda) = T^n \times P^n \big/ \sim \tag{3.2}$$

is a smooth, closed, connected, $2n$-dimensional manifold with T^n action induced by left translation [7, page 423]. A projection $\pi\colon M^{2n} \to P^n$ onto the polytope is induced from the projection $T^n \times P^n \to P^n$.

Remark. In the cases when M^{2n} is a projective non-singular toric variety, P^n and λ encode the information in the defining fan.

Suppose that K is dual to a simple polytope P^n having m facets. Recall that the duality here is in the sense that the facets of P^n correspond to the vertices of K. A set of vertices in K is a simplex if and only if the corresponding facets in P^n all intersect. At each vertex of a simple polytope P^n, exactly n facets intersect.

A characteristic function $\lambda\colon \mathcal{F} \longrightarrow \mathbb{Z}^n$, assigns an integer vector to each facet of the simple polytope P^n. It can be considered as an $(n \times m)$-matrix, $\lambda\colon \mathbb{Z}^m \longrightarrow \mathbb{Z}^n$, with integer entries and columns indexed by the facets of P^n. The condition following (3.1) may be interpreted as requiring all $n \times n$ minors of λ, corresponding to the vertices of P^n, to be ± 1.

Given λ and $J = (j_1, j_2, \ldots, j_m)$, a new function

$$\lambda(J)\colon \mathbb{Z}^{d(J)} \longrightarrow \mathbb{Z}^{d(J)-m+n}$$

can be constructed by taking $\lambda(J)$ to be the $\big(\big(d(J) - m + n\big) \times d(J)\big)$-matrix described in Figure 1 below. In the diagram, the columns of the matrix are indexed by the vertices of $K(J)$ and I_k denotes a $k \times k$ identity sub-matrix.

The next theorem constructs an infinite family of toric manifolds "derived" from the information in λ, P^n and $J = (j_1, j_2, \ldots, j_m)$.

Theorem 3.1. *If λ is a characteristic map for a $2n$-dimensional toric manifold M, then $\lambda(J)$ is the characteristic map for a toric manifold $M(J)$ of dimension $2d(J) - 2m + 2n$.*

Proof. Theorem 2.4 ensures that $P(J)$ is a simple polytope of dimension $d(J) - m + n$. It remains to show that for each vertex of $P(J)$, the corresponding $\big(d(J)-m+n\big) \times \big(d(J)-m+n\big)$ minor of $\lambda(J)$ is equal to ± 1.

$$
\begin{array}{c|c|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{\boldsymbol{v_{12}\cdots\ v_{1j_1}}} & \multicolumn{1}{c}{\boldsymbol{v_{22}\cdots\ v_{2j_2}}} & \multicolumn{1}{c}{\cdots} & \multicolumn{1}{c}{\boldsymbol{v_{m2}\cdots v_{mj_m}}} & \multicolumn{1}{c}{\boldsymbol{v_{11}\ v_{21}\ \cdots\ v_{m1}}} \\
\cline{2-6}
& I_{j_1-1} & 0 & \cdots & 0 & \begin{matrix} -1 & \\ -1 & \\ \vdots & 0 \\ -1 & \end{matrix} \\
\cline{2-6}
& 0 & I_{j_2-1} & 0 & 0 & \begin{matrix} 0 & -1 & \\ 0 & -1 & \\ \vdots & \vdots & 0 \\ 0 & -1 & \end{matrix} \\
\cline{2-6}
& \vdots & 0 & \ddots & 0 & \vdots \\
\cline{2-6}
& 0 & \vdots & 0 & I_{j_m-1} & \begin{matrix} & -1 \\ & -1 \\ 0 & \vdots \\ & -1 \end{matrix} \\
\cline{2-6}
& 0 & 0 & 0 & 0 & \lambda \\
\cline{2-6}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\boldsymbol{1\ \ 2\ \ \cdots\ \ m}}
\end{array}
\quad
\begin{matrix} \boldsymbol{1} \\ \boldsymbol{2} \\ \vdots \\ \boldsymbol{n} \end{matrix}
$$

Figure 1. The matrix $\lambda(J)$

The proof is by induction. Consider first the case $J = (1, 1, \ldots, 1, \overset{i}{2}, 1, \ldots, 1)$. Corresponding to the re-indexing of the vertices of $K(J)$, the facets of $P(J)$ are indexed as follows

$$\mathcal{F}(J) = \{F_{11}, F_{21}, \ldots, F_{(i-1)1}, F_{i1}, F_{i2}, F_{(i+1)1}, \ldots, F_{m1}\}$$

The matrix $\lambda(J)$ now has the form

$$\begin{array}{cccccc} F_{i2} & F_{11} & & F_{i1} & & F_{m1} \end{array}$$
$$\begin{pmatrix} 1 & 0 & \cdots & -1 & 0 & 0 \\ 0 & \lambda_{11} & \cdots & \lambda_{1i} & \cdots & \lambda_{1m} \\ 0 & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \lambda_{n1} & \cdots & \lambda_{ni} & \cdots & \lambda_{nm} \end{pmatrix}$$

where $\lambda = (\lambda_{ij})$ is the original matrix. (Recall that vertices v_{ik} of $K(J)$ correspond to facets F_{ik} of the polytope $P(J)$.) The minors corresponding to the new $(n+1)$-fold intersections of facets, are of two types.

(1) Those which include columns indexed by both F_{i1} and F_{i2}.
(2) Those which include columns indexed by either F_{i1} or F_{i2} but not both.

This observation follows from the fact that the simplicial wedge construction ensures that each new maximal simplex of $K(J) = K(v_i)$ must contain either v_{i1} or v_{i2}. According to the discussion following Construction 2.2, the first type arise from intersections

$$F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k} \cap F_i \cap F_{i_{k+1}} \cap \cdots \cap F_{i_n} \tag{3.3}$$

of n facets in P^n. In $P(J)$, they give $(n+1)$-fold intersections

$$F_{i_1 1} \cap F_{i_2 1} \cap \cdots \cap F_{i_k 1} \cap F_{i1} \cap F_{i2} \cap F_{i_{(k+1)} 1} \cap \cdots \cap F_{i_n 1}.$$

The corresponding $(n+1) \times (n+1)$ minors in the matrix $\lambda(J)$ above are

$$\begin{array}{cccccccc}
F_{i2} & F_{i_11} & & F_{i_k1} & F_{i1} & F_{i_{k+1}1} & & F_{i_n1} \\
\end{array}$$
$$\begin{vmatrix}
1 & 0 & \cdots & 0 & -1 & 0 & 0 & 0 \\
0 & \lambda_{1i_1} & \cdots & \lambda_{1i_k} & \lambda_{1i} & \lambda_{1i_{k+1}} & \cdots & \lambda_{1i_n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & \lambda_{ni_1} & \cdots & \lambda_{ni_k} & \lambda_{ni} & \lambda_{1i_{k+1}} & \cdots & \lambda_{ni_n}
\end{vmatrix}$$

Expanding by the first row gives ± 1 by (3.3). Now $(n+1)$-fold intersections of facets of the second type, which contain F_{i2} but not F_{i1} (or vice versa) arise from intersections in P which do not involve the facet F_i. That is, they are of the form

$$F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k} \cap F_{i_{k+1}} \cap \cdots \cap F_{i_n}, \quad i_j \neq i \tag{3.4}$$

In $P(J)$ the intersection is

$$F_{12} \cap F_{i_11} \cap F_{i_21} \cap \cdots \cap F_{i_k1} \cap F_{i_{(k+1)1}} \cap \cdots \cap F_{i_n1}, \quad i_j \neq i.$$

The $(n+1) \times (n+1)$ minor in $\lambda(J)$ will have the form

$$\begin{array}{cccc}
F_{i2} & F_{i_11} & & F_{i_n1} \\
\end{array}$$
$$\begin{vmatrix}
1 & 0 & \cdots & 0 \\
0 & \lambda_{1i_1} & \cdots & \lambda_{1i_n} \\
\vdots & \vdots & & \vdots \\
0 & \lambda_{ni_1} & \cdots & \lambda_{ni_n}
\end{vmatrix}$$

and so have the value ± 1. Finally, $(n+1)$-fold intersections of facets of the second type, which contain F_{i1} but not F_{i2} again arise from intersections of the type 3.4. In $P(J)$ the intersection is

$$F_{i_11} \cap F_{i_21} \cap \cdots \cap F_{i_k1} \cap F_{i1} \cap F_{i_{(k+1)1}} \cap \cdots \cap F_{i_n1}.$$

and the corresponding $(n+1) \times (n+1)$ minor in $\lambda(J)$ has the form

$$\begin{array}{c}
\begin{array}{ccccccc} F_{i_1 1} & & F_{i_k 1} & F_{i1} & F_{i_{k+1}1} & & F_{i_n 1} \end{array} \\
\left|\begin{array}{ccccccc}
0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\lambda_{1i_1} & \cdots & \lambda_{1i_k} & \lambda_{1i} & \lambda_{1i_{k+1}} & \cdots & \lambda_{1i_n} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\lambda_{ni_1} & \cdots & \lambda_{ni_k} & \lambda_{ni} & \lambda_{1i_{k+1}} & \cdots & \lambda_{ni_n}
\end{array}\right|
\end{array}$$

Expansion by the first row gives the $n \times n$ minor in λ

$$\left|\begin{array}{cccccc}
\lambda_{1i_1} & \cdots & \lambda_{1i_k} & \lambda_{1i_{k+1}} & \cdots & \lambda_{1i_n} \\
\vdots & & \vdots & \vdots & & \vdots \\
\lambda_{ni_1} & \cdots & \lambda_{ni_k} & \lambda_{1i_{k+1}} & \cdots & \lambda_{ni_n}
\end{array}\right|$$

which has the value ± 1 because it corresponds to (3.4). The inductive step passes from the m-tuple $J = (j_1, j_2, \ldots, j_m)$ to $J' = (j_1, j_2, \ldots, j_{k-1}, j_k + 1, j_{k+1} \ldots, j_m)$ and follows the same argument, replacing the characteristic map λ in the discussion above with $\lambda(J)$, This completes the proof. $\square$

Remark 3.2. Work in progress by the authors will show that the manifolds $M(J)$ can be obtained alternatively by a reinterpretation and generalization of the Davis-Januszkiewicz construction (3.2).

4. The cohomology of the toric manifolds $M(J)$

The rows of the matrix $\lambda(J)$ determine an ideal $L_{M(J)}$ generated by linear relations among the generators of the Stanley-Reisner ring of $K(J)$. These are given by:

$$v_{it} - v_{i1} = 0, \quad t = 2, \ldots, j_i, \quad i = 1, \ldots, m$$

(4.1)

$$\lambda_{i1}v_{11} + \lambda_{i2}v_{21} + \ldots + \lambda_{im}v_{m1} = 0, \quad i = 1, \ldots, n$$

Notice that the second set of relations are those corresponding to the linear ideal determined by the matrix λ. The next result is the Davis-Januszkiewicz, (Danilov-Jurkewicz) theorem, [7], for the toric manifold $M(J)$.

Theorem 4.1. *The cohomology ring $H^*(M(J);\mathbb{Z})$ is isomorphic to*

$$\mathbb{Z}[v_{11}, v_{12}, \ldots, v_{1j_1}, v_{21}, v_{22}, \ldots, v_{2j_2},\ \ldots\ , v_{m1}, v_{m2}, \ldots, v_{mj_m}]\Big/\big(I_{K(J)} + L_{M(J)}\big)$$

where $I_{K(J)}$ denotes the Stanley-Reisner ideal for the simplicial complex $K(J)$. □

Applying the linear relations (4.1) and rewriting v_{i1} as v_i, allows a significant simplification of this description.

Theorem 4.2. *The cohomology ring $H^*(M(J);\mathbb{Z})$ is isomorphic to*

$$\mathbb{Z}[v_1, v_2, \ldots, v_m]/(I_K^J + L_M)$$

where each v_i has degree two, L_M is the ideal in the Stanley-Reisner ring of K generated by the rows of the matrix λ and I_K^J is the ideal of relations generated by all monomials of the form

$$v_{i_1}^{j_{i_1}} v_{i_2}^{j_{i_2}} \cdots v_{i_k}^{j_{i_k}} \tag{4.2}$$

corresponding to the minimal non-simplex $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ of K.

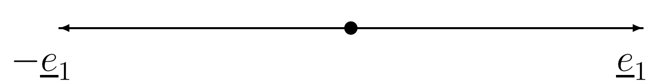
Proof. Linear relations (4.1) and the relabeling of v_{i1} as v_i, convert the monomials generating $I_{K(J)}$ into those of (4.2) and, the relations $L_{M(J)}$ into the relations L_M. □

5. The example of complex projective space

As a toric manifold, $\mathbb{C}P^1$ is constructed from the polytope $P^1 = \Delta^1$ which is the one-simplex. The dual simplicial complex is $K = \big\{\{v_1\}, \{v_2\}\big\}$ consisting of two disjoint vertices. Here, $m = 2$ and $n = 1$. The characteristic matrix $\lambda\colon \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ is the matrix

$$\lambda = \begin{pmatrix} -1 & 1 \end{pmatrix}$$

which corresponds to the one-dimensional fan



Choosing $J = (k, 1)$ produces the $k \times (k+1)$ matrix

$$\begin{pmatrix} & & & -1 & 0 \\ & \boxed{I_{k-1}} & & \vdots & 0 \\ & & & -1 & \vdots \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

which corresponds to the characteristic function (indeed, fan) of the complex projective space $\mathbb{C}P^k$. Recall now that the Stanley-Reisner ring of the simplicial complex $K = \big\{\{v_1\}, \{v_2\}\big\}$ is

$$\mathbb{Z}[v_1, v_2]/\langle v_1 v_2\rangle$$

The description of $H^*(\mathbb{C}P^k; \mathbb{Z})$ given by Theorem 4.2, becomes

$$\mathbb{Z}[v_1, v_2]\Big/\big(\langle v_1^k v_2\rangle, v_1 = v_2\big)$$

which translates into $H^*(\mathbb{C}P^k; \mathbb{Z}) \cong \mathbb{Z}[v]/v^{k+1}$, as expected. This is to be compared with the usual calculation from the Danilov-Jurkewicz theorem, (Theorem 4.1) which is

$$H^*(\mathbb{C}P^k; \mathbb{Z}) \cong \mathbb{Z}[v_1, v_2, \ldots, v_{k+1}]\Big/\big(\langle v_1 v_2 \cdots v_{k+1}\rangle, v_1 = v_2 = \ldots = v_{k+1}\big).$$

Remark. Notice that taking $J = (j_1, j_2)$, satisfying $j_1 + j_2 = k + 1$, yields $\mathbb{C}P^1(J) = \mathbb{C}P^k$ also.

6. Polyhedral products

Let K be a simplicial complex with m vertices and let $(\underline{X}, \underline{A})$ denote a family of CW pairs

$$(X_1, A_1), (X_2, A_2), \ldots, (X_m, A_m).$$

When all the pairs (X_i, A_i) are the same pair (X, A), the family $(\underline{X}, \underline{A})$ is written simply as (X, A). A polyhedral product, is a topological space

$$Z(K; (\underline{X}, \underline{A})) \subseteq \prod_{i=1}^{m} X_i\ ,$$

defined as a colimit by a diagram $D : K \to CW_*$. At each $\sigma \in K$, it is given by

$$D(\sigma) = \prod_{i=1}^{m} W_i, \quad \text{where} \quad W_i = \begin{cases} X_i & \text{if} \quad i \in \sigma \\ A_i & \text{if} \quad i \in [m] - \sigma. \end{cases} \tag{6.1}$$

Here, the colimit is a union given by

$$Z(K; (\underline{X}, \underline{A})) = \bigcup_{\sigma \in K} D(\sigma).$$

Detailed background information about polyhedral products may be found in [5], [6], [1], [2] (and in the unpublished notes of N. Strickland).

The family of CW-pairs $(\underline{X}, \underline{A})$ to be investigated here, is specified by the sequence of positive integers $J = (j_1, \dots, j_m)$ and is given by

$$(\underline{X}, \underline{A}) \;=\; (\underline{D}^{2J}, \underline{S}^{2J-1}) \;=\; \big\{(D^{2j_i}, S^{2j_i-1})\big\}_{i=1}^{m} \tag{6.2}$$

(Presently, it will become necessary to consider the discs D^{2j_i} as embedded naturally in $\mathbb{C}^{j_i}$.) If $J = (1, 1, \dots, 1)$, the space $Z(K; (\underline{D}^{2J}, \underline{S}^{2J-1}))$ is an ordinary moment-angle complex and is written $Z(K; (D^2, S^1))$.

The spaces $Z(K; (\underline{D}^{2J}, \underline{S}^{2J-1}))$ all have the property of being *stably wedge equivalent*.

Definition 6.1. Two spaces X and Y are said to be stably wedge equivalent if there are stable homotopy equivalences

$$X \sim X_1 \vee X_2 \vee \ldots \vee X_t \quad \text{and} \quad Y \sim Y_1 \vee Y_2 \vee \ldots \vee Y_t$$

and X_i is stably homotopy equivalent to Y_i for all $i = 1, 2, \dots, t$.

The next proposition follows directly from the stable splitting theorems for generalized moment-angle complexes of [1] and [2, Corollary 2.24].

Proposition 6.2. *For all $J = (j_1, \dots, j_m)$, the spaces $Z(K; (\underline{D}^{2J}, \underline{S}^{2J-1}))$ are all stably wedge equivalent and moreover, they have isomorphic ungraded cohomology rings.* $\square$

7. The simplicial wedge construction and polyhedral products

Recall that a *product* of CW pairs is defined by

$$(X, A) \times (Y, B) := \big(X \times Y, (X \times B) \cup (A \times Y)\big). \tag{7.1}$$

The k-fold iteration $(X, A) \times \cdots \times (X, A)$ is denoted by $(X, A)^k$.

Let K be a simplicial complex on m vertices $\{v_1, v_2, \ldots, v_m\}$ and let $(\underline{X}, \underline{A})$ denote the family of CW pairs

$$(X_1, A_1), (X_2, A_2), \ldots, (X_m, A_m).$$

In the light of Definition 2.1, it becomes necessary at this point to introduce a notational convention to avoid expressions becoming too unwieldy.

Convention. Let $J = (j_1, j_2, \ldots, j_m)$ be sequence of positive integers, $K(J)$ as in Definition 2.1 and the family of pairs $(\underline{X}, \underline{A})$ as above. Denote by $Z\big(K(J); (\underline{X}, \underline{A})\big)$ the polyhedral product determined by the simplicial complex $K(J)$ and the family of pairs obtained from $(\underline{X}, \underline{A})$ by repeating each (X_i, A_i), j_i times in sequence.

Fix $i \in \{1, 2, \ldots, m\}$ and define a family of CW pairs $(\underline{Y}, \underline{B})$ by

$$(Y_k, B_k) = \begin{cases} (X_k, A_k) & \text{if} \quad k \neq i \\ (X_i, A_i)^2 & \text{if} \quad k = i. \end{cases}$$

Theorem 7.1. *The polyhedral product spaces $Z(K; (\underline{Y}, \underline{B}))$ and $Z\big(K(v_i); (\underline{X}, \underline{A})\big)$ are equivalent subspaces of $W_1 \times \cdots \times W_m$, where*

$$W_k = \begin{cases} X_k & \text{if} \quad k \neq i \\ X_k \times X_k, & \text{if} \quad k = i. \end{cases}$$

Proof. As in Construction 2.2, the vertices of $K(v_i)$ are

$$S = \{v_{11}, \ldots, v_{(i-1)1}, v_{i1}, v_{i2}, v_{(i+1)1}, \ldots, v_{m1}\}.$$

Let $\sigma = \{v_{t_1}, v_{t_2}, \ldots, v_{t_k}\}$ be a maximal simplex in K. If $v_i = v_{t_s} \in \sigma$, then $D(\sigma)$ is identified by the identity map with $D(\sigma') \subset Z(K; (\underline{Y}, \underline{B}))$ where

$$\sigma' = \{v_{t_1 1}, v_{t_2 1}, \ldots, v_{(t_s-1)1}, v_{t_s 1}, v_{t_s 2}, v_{(t_s+1)1}, \ldots, v_{t_k 1}\}.$$

If $v_i \notin \sigma$, then $D(\sigma)$ is identified by the identity map with $D(\sigma_1') \cup D(\sigma_2') \subset Z(K; (\underline{Y}, \underline{B}))$, where

$$\begin{aligned} \sigma_1' &= \{v_{t_1 1}, v_{t_2 1}, \ldots, v_{t_k 1}, v_{i1}\}. \\ \sigma_2' &= \{v_{t_1 1}, v_{t_2 1}, \ldots, v_{t_k 1}, v_{i2}\}. \end{aligned}$$

(Here, the vertices may not be in their correct order.) The fact that the wedge construction ensures that the maximal simplices σ_1' and σ_2' exist in $K(v_i)$ has been used here. This procedure exhausts all maximal simplices in $K(v_i)$ and completes the equivalence. □

The next iteration of this procedure is slightly more involved but serves to describe all iterations. Choose $j \in \{1, 2, \ldots, m\}$. If $j \neq i$, the new family $(\underline{Y}, \underline{B})$ is defined by

$$(Y_k, B_k) = \begin{cases} (X_k, A_k) & \text{if} \quad k \neq i, j \\ (X_k, A_k)^2 & \text{if} \quad k = i, j. \end{cases}$$

In this case, $K(v_i)$ is replaced with $\big(K(v_i)\big)(v_{j1})$ and the procedure is exactly as described above. The result is that $Z(K; (\underline{Y}, \underline{B}))$ and $Z\big((K(v_i))(v_{j1}); (\underline{X}, \underline{A})\big)$ are equivalent subspaces of $W_1 \times \cdots \times W_m$, where

$$W_k = \begin{cases} X_k & \text{if} \quad k \neq i, j \\ X_k \times X_k, & \text{if} \quad k = i, j. \end{cases}$$

In the case $j = i$, the new family $(\underline{Y}, \underline{B})$ will have

$$(Y_k, B_k) = \begin{cases} (X_k, A_k) & \text{if} \quad k \neq i \\ (X_i, A_i)^3 & \text{if} \quad k = i. \end{cases}$$

In this case, $K(v_i)$ is replaced with either $\big(K(v_i)\big)(v_{i1})$ or $\big(K(v_i)\big)(v_{i2})$. The symmetry of the wedge construction ensures that these two simplicial complexes are isomorphic as simplicial complexes. The result now is that $Z(K; (\underline{Y}, \underline{B}))$ and $Z\big((K(v_i))(v_{i1}); (\underline{X}, \underline{A})\big)$ are equivalent subspaces of $W_1 \times \cdots \times W_m$, where

$$W_k = \begin{cases} X_k & \text{if} \quad k \neq i \\ \prod_{s=1}^{3} X_k & \text{if} \quad k = i. \end{cases}$$

The general iteration procedure is now straightforward; the result is recorded in the next theorem.

Theorem 7.2. *Let K be a simplicial complex with m vertices and let $(\underline{X}, \underline{A})$ denote a family of CW pairs*

$$\{(X_1, A_1), (X_2, A_2), \ldots, (X_m, A_m)\}.$$

For $J = (j_1, j_2, \ldots, j_m)$, a sequence of positive integers, define a new family of pairs $(\underline{Y}, \underline{B}))$ by

$$(Y_k, B_k) \ = \ (X_k, A_k)^{j_k}, \quad k = 1, 2, \ldots, m.$$

Then, the polyhedral product spaces $Z(K; (\underline{Y}, \underline{B}))$ and $Z\big(K(J); (\underline{X}, \underline{A})\big)$ are equivalent subspaces of $X_1^{j_1} \times X_2^{j_2} \times \cdots \times X_m^{j_m}$. □

This result is applied to the family where $(X_i, A_i) = (D^2, S^1)$ for all $i = 1, 2, \ldots, m$. In this case,

$$(Y_k, B_k) \ = \ (X_k, A_k)^{j_k} \ = \ \big((D^2)^{j_k}, \partial\big((D^2)^{j_k}\big)\big),$$

where $\partial\big((D^2)^{j_i}\big)$ denotes the boundary of a j_i-fold product of two-discs. For this particular case, the family $(\underline{Y}, \underline{B})$ is denoted by

$$(\underline{B}^{2J}, \underline{\partial B}^{2J}) := \ \big\{(D^2)^{j_i}, \partial\big((D^2)^{j_i}\big)\big\}_{i=1}^{m}.$$

Theorem 7.2 implies now the next corollary.

Corollary 7.3. *The generalized moment-angle complex $Z(K; (\underline{B}^{2J}, \underline{\partial B}^{2J}))$ and the moment-angle complex $Z\big(K(J); (D^2, S^1)\big)$ are equivalent subspaces of*

$$(D^2)^{j_1} \times (D^2)^{j_2} \times \cdots \times (D^2)^{j_m} = (D^2)^{d(J)}.$$

Remark. Notice that by considering $(D^2)^{d(J)} \subset \mathbb{C}^{d(J)}$, both moment-angle complexes inherit an action of the real torus $T^{d(J)}$ with respect to which they are equivariantly equivalent.

An entirely similar argument shows that taking $(X_i, A_i) = (D^1, S^0)$ for all $i = 1, 2, \ldots, m$ and $J = (2, 2, \ldots, 2)$, yields the result that $Z\big(K; (D^1 \times D^1, \partial(D^1 \times D^1)\big)$ and $Z(K(J); (D^1, S^0))$ are equivalent subspaces of $(D^1)^2 \times (D^1)^2 \times \cdots \times (D^1)^2$. It follows by the arguments below that $Z(K; (D^2, S^1))$ and $Z(K(J); (D^1, S^0))$ are diffeomorphic.

Recall now the family of pairs $(\underline{D}^{2J}, \underline{S}^{2J-1})$ described in (6.2). Observe that for the corresponding generalized moment-angle complex, there is a natural embedding,

$$Z(K; (\underline{D}^{2J}, \underline{S}^{2J-1})) \;\subseteq\; D^{2j_1} \times D^{2j_2} \times \cdots \times D^{2j_m}$$

The next goal is to verify that the generalized moment-angle complexes $Z(K; (\underline{B}^{2J}, \underline{\partial B}^{2J}))$ and $Z(K; (\underline{D}^{2J}, \underline{S}^{2J-1}))$ are equivariantly diffeomorphic with respect to various torus actions.

Let $(D^2)^{j_i} \subset \mathbb{C}^{j_i}$ be embedded in the usual way and choose a standard diffeomorphism $h_i \colon (D^2)^{j_i} \longrightarrow D^{2j_i}$. Define an action of the circle T^1 on D^{2j_i} by

$$t \cdot h_i(z_1, z_2, \ldots, z_{j_i}) \;=\; h_i(tz_1, tz_2, \ldots, tz_{j_i}).$$

Next, denote the j_i-tuple $(z_1, z_2, \ldots, z_{j_i})$ by the symbol $\underline{z}_{j_i}$ and define an action of T^m on $(D^2)^{d(J)}$ by

$$(t_1, t_2, \ldots, t_m) \cdot (\underline{z}_{j_1}, \underline{z}_{j_2}, \ldots, \underline{z}_{j_m}) \;=\; (t_1\underline{z}_{j_1}, t_2\underline{z}_{j_2}, \ldots, t_m\underline{z}_{j_m}) \tag{7.2}$$

where $t_i\underline{z}_{j_i}$ has the usual meaning $(t_iz_1, t_iz_2, \ldots, t_iz_{j_i})$. Notice that this action of T^m is a restriction of the natural action of the torus $T^{d(J)}$ on $(D^2)^{d(J)}$. An action of T^m is induced on $D^{2j_1} \times D^{2j_2} \times \cdots \times D^{2j_m}$ by

$$(t_1, t_2, \ldots, t_m) \cdot \big(h_1(\underline{z}_{j_1}), h_2(\underline{z}_{j_2}), \ldots, h_m(\underline{z}_{j_m})\big) = \big(h_1(t_1\underline{z}_{j_1}), h_2(t_2\underline{z}_{j_2}), \ldots, h_m(t_m\underline{z}_{j_m})\big). \tag{7.3}$$

The diffeomorphisms h_i give rise to a diffeomorphism, equivariant with respect to the T^m-action above,

$$H \colon (D^2)^{d(J)} \longrightarrow D^{2j_1} \times D^{2j_2} \times \cdots \times D^{2j_m} \tag{7.4}$$

by $H(\underline{z}_{j_1}, \underline{z}_{j_2}, \ldots, \underline{z}_{j_m}) = \big(h_1(\underline{z}_{j_1}), h_2(\underline{z}_{j_2}), \ldots, h_m(\underline{z}_{j_m})\big)$. The diffeomorphism H restricts to a diffeomorphism of generalized moment-angle complexes.

Lemma 7.4. *The diffeomorphism H restricts to a diffeomorphism of generalized moment-angle complexes $Z(K;(\underline{B}^{2J},\underline{\partial B}^{2J}))$ and $Z(K;(\underline{D}^{2J},\underline{S}^{2J-1}))$, which is equivariant with respect to the T^m-action defined by (7.3).* $\square$

Combining this observation with Corollary 7.3 and the remark following it, yields a key result.

Theorem 7.5. *There is an action of T^m on both $Z(K;(\underline{D}^{2J},\underline{S}^{2J-1}))$ and $Z\big(K(J);(D^2,S^1)\big)$, with respect to which they are equivariantly diffeomorphic.* $\square$

When combined with Theorem 1.8 of [3], this theorem yields an immediate corollary.

Corollary 7.6. *The spaces $Z(K;(D^2,S^1))$ and $Z\big(K(J);(D^2,S^1)\big)$ have isomorphic ungraded cohomology rings.* $\square$

These results yield an observation about the action of the Steenrod algebra.

Corollary 7.7. *There is an isomorphism of ungraded $\mathbb{Z}/2$-modules*

$$H^*\big(Z(K;(D^2,S^1));\mathbb{Z}/2\big) \longrightarrow H^*\big(Z\big(K(J);(D^2,S^1)\big);\mathbb{Z}/2\big)$$

which commutes with the action of the Steenrod algebra.

Proof. The Steenrod operations are stable operations and hence the splitting theorem [2, Theorem 2.21] implies that there is an isomorphism of ungraded $\mathbb{Z}/2$-modules

$$H^*\big(Z(K;(D^2,S^1));\mathbb{Z}/2\big) \longrightarrow H^*\big(Z(K;(\underline{D}^{2J},\underline{S}^{2J-1}));\mathbb{Z}/2\big)$$

which commutes with the action of the Steenrod algebra. The result follows from Theorem 7.5. $\square$

Remark. Corollary 7.7 holds equally well for $\mathbb{Z}/p$ with p an odd prime.

8. A generalization to topological joins

As usual, let K be a simplicial complex on m vertices and $J=(j_1,j_2,\ldots,j_m)$, a sequence of positive integers. Consider the family of pairs

$$(\underline{CX},\underline{X}) \;=\; \big\{(CX_i,X_i)\big\}_{i=1}^{m}$$

where, each space X_i is a CW complex and CX_i denotes the cone on X_i. Applying Theorem 7.2 to this family of pairs yields an equivalence of polyhedral products

$$Z(K;(\underline{Y},\underline{B})) \longrightarrow Z\big(K(J);(\underline{CX},\underline{X})\big) \tag{8.1}$$

where $(Y_k,B_k) \;=\; (CX_k,X_k)^{j_k},\; k=1,2,\ldots,m.$

The equivalence is as subspaces of $(CX_1)^{j_1}\times(CX_2)^{j_2}\times\cdots\times(CX_m)^{j_m}$. The next lemma, which is well known, is a key ingredient.

Lemma 8.1. *For any finite CW complex X, there is a homeomorphism of pairs*

$$\big(C(X*X),X*X\big)\xrightarrow{f}(CX,X)^2,$$

where $*$ denotes the topological join.

Proof. Represent a point in $C(X*X)$ by $\big[s,[x_1,t,x_2]\big]$. Define the homeomorphism f by

$$f\big(\big[s,[x_1,t,x_2]\big]\big) \;=\; \big([2s\cdot\min\{t,1/2\},x_1],[2s\cdot\min\{1-t,1/2\},x_2]\big)\in CX\times CX,$$

where the cone point is at $s=0$. At $s=1$, f is the usual homeomorphism

$$X*X\longrightarrow(CX\times X)\cup(X\times CX).$$

The map f is a continuous bijection between compact Hausdorff spaces and hence is a homeomorphism. ☐

Next, define a family of CW pairs by

$$\big(\underline{C(\circledast_J X)},\;\underline{\circledast_J X}\big)\;:=\;\big\{\big(C(\underbrace{X_i*X_i*\cdots*X_i}_{j_i}),\underbrace{X_i*X_i*\cdots*X_i}_{j_i}\big)\big\}_{i=1}^{m}.$$

The map f of Lemma 8.1 iterates easily to produce a homeomorphism of pairs

$$\big(C(\underbrace{X_k*X_k*\cdots*X_k}_{j_k}),\underbrace{X_k*X_k*\cdots*X_k}_{j_k}\big)\xrightarrow{f_{j_k}}(CX_k,X_k)^{j_k},$$

which extends to a map of families of pairs,

$$\big(\underline{C(\circledast_J X)},\ \underline{\circledast_J X}\big) \xrightarrow{f_J} (\underline{Y}, \underline{B})).$$

The results above combine now to give the next theorem.

Theorem 8.2. *There is a homeomorphism of polyhedral products,*

$$Z\Big(K; \big(\underline{C(\circledast_J X)},\ \underline{\circledast_J X}\big)\Big) \longrightarrow Z\big(K(J); (\underline{CX}, \underline{X})\big).$$

Proof. The homeomorphism is given by composing the homeomorphism of polyhedral products induced by f_J with the homeomorphism of (8.1). □

9. Toric manifolds and generalized moment-angle complexes

The information in a toric manifold M^{2n} can be recorded concisely as a triple (P^n, λ, M^{2n}). Let K be the simplicial complex dual to the polytope P^n. The moment-angle complex $Z(K; (D^2, S^1))$ is a subcomplex of the product of two-discs

$$Z(K; (D^2, S^1)) \subseteq (D^2)^m \subset \mathbb{C}^m.$$

As such, it has a natural action of the real m-torus T^m. If λ satisfies the condition following (3.1), then the kernel of λ, a subgroup $T^{m-n} \subset T^m$, acts freely on $Z(K; (D^2, S^1))$ and the quotient is homeomorphic to M^{2n}, [7, page 434], and [6, page 86]. The action of T^n on M^{2n}, which yields P^n as orbit space, is that given by the action of the quotient T^m/T^{m-n} on $Z(K; (D^2, S^1))/T^{m-n}$.

The freeness of the action of T^{m-n} on $Z(K; (D^2, S^1))$ implies a homotopy equivalence of Borel spaces

$$ET^m \times_{T^m} Z(K; (D^2, S^1)) \longrightarrow ET^n \times_{T^n} M^{2n}. \tag{9.1}$$

Moreover, for any simplicial complex K, there is an equivalence ([6]),

$$ET^m \times_{T^m} Z(K; (D^2, S^1)) \;\cong\; Z(K; (\mathbb{C}P^\infty, *)), \tag{9.2}$$

where the right hand side is a polyhedral product which is a subcomplex of the product space $(\mathbb{C}P^\infty)^m$. These spaces are called, (loosely), the Davis-Januszkiewicz spaces associated to the simplicial complex K and are denoted by the symbol $\mathcal{DJ}(K)$. Also, the cohomology ring $H^*(\mathcal{DJ}(K); \mathbb{Z})$ is the Stanley-Reisner ring of the simplicial complex K. The Serre spectral sequence of the fibration

$$M^{2n} \longrightarrow ET^n \times_{T^n} M^{2n} \xrightarrow{\;p\;} BT^n \tag{9.3}$$

yields an entirely topological computation of the ring $H^*(M; \mathbb{Z})$. Known as the Davis-Januszkiewicz theorem, it generalizes the Danilov-Jurkewicz theorem for projective non-singular toric varieties, [7]. Applied to the manifolds $M(J)$, it gives Theorem 4.1.

Given (P^n, λ, M^{2n}), let $\lambda(J)$ be the matrix defined by Figure 1. Choosing a standard basis and following [6], the kernel of λ, as a sub-torus of T^m, is specified by an $m \times (m-n)$-matrix $S = (s_{ij})$. Explicitly, it is given by

$$\ker\lambda \;=\; \left\{\left(e^{2\pi i(s_{11}\phi_1 + \cdots + s_{1(m-n)}\phi_{m-n})}, \ldots, e^{2\pi i(s_{m1}\phi_1 + \cdots + s_{m(m-n)}\phi_{m-n})}\right) \in T^m\right\}$$

where $\phi_i \in \mathbb{R}, i = 1, 2, \ldots, m-n$. The form of the matrix in Figure 1 reveals the kernel of $\lambda(J)$ to be specified by the $d(J) \times (m-n)$-matrix $S(J) = (s^J_{lk})$, where

$$s^J_{j_ik} = s_{ik}, \quad i = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, m-n. \tag{9.4}$$

Notice that this description makes explicit the isomorphism

$$\ker\lambda(J) \;\cong\; T^{m-n}.$$

The action of $\ker\lambda(J)$ on both $Z(K; (\underline{D}^{2J}, \underline{S}^{2J-1}))$ and $Z\big(K(J); (D^2, S^1)\big)$ is via the inclusions

$$\ker\lambda(J) \;\subset\; T^m \;\subset\; T^{d(J)},$$

where the first inclusion is determined by the matrix S and (9.4); the second is determined by (7.3). Now, ker $\lambda(J)$ acts freely on $Z\big(K(J);(D^2,S^1)\big)$ by Theorem 3.1 and hence on $Z(K;(\underline{D}^{2J},\underline{S}^{2J-1}))$ by Theorem 7.5 which implies also the result following.

Lemma 9.1. *There is a diffeomorphisms of orbit spaces*

$$Z\big(K(J);(D^2,S^1))\big)/\text{ker }\lambda(J) \longrightarrow Z(K;(\underline{D}^{2J},\underline{S}^{2J-1}))/\ker\lambda(J).$$

Finally, Theorem 3.1 and the discussion at the beginning of this section imply that $M(J)$ is diffeomorphic to $Z\big(K(J);(D^2,S^1))\big)/\text{ker }\lambda(J)$. The main theorem follows now from Lemma 9.1.

Theorem 9.2. *There is a diffeomorphism*

$$M(J) \longrightarrow Z(K;(\underline{D}^{2J},\underline{S}^{2J-1}))/\ker\lambda(J).$$

10. The analogue of the Davis-Januszkiewicz space

The fibration (9.3) is used in the standard theory to exhibit $H^*(M^{2n};\mathbb{Z})$ as a quotient of the cohomology of the Davis-Januszkiewicz space. In its various guises, this space is

$$\mathcal{DJ}(K) := ET^n \times_{T^n} M^{2n} \;\simeq\; ET^m \times_{T^m} Z(K;(D^2,S^1)) \;\simeq\; Z(K;(\mathbb{C}P^\infty,*)).$$

The recognition of $M(J)$ as the quotient $Z(K(J);(D^2,S^1))/\ker\lambda(J)$, yields the cohomology calculation in Theorem 4.1. To get the more concise calculation afforded by Theorem 4.2, the description of $M(J)$ given by Theorem 9.2 is used instead. So, an appropriate analogue of the space $\mathcal{DJ}(K)$ is needed.

Define a family of CW pairs by

$$(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{J-1}) := \big\{(\mathbb{C}P^\infty,\mathbb{C}P^{j_1-1}),\ldots,(\mathbb{C}P^\infty,\mathbb{C}P^{j_m-1})\big\}$$

and consider the polyhedral product

$$Z(K; (\underline{\mathbb{C}P}^{\infty}, \underline{\mathbb{C}P}^{J-1})) \subseteq (\mathbb{C}P^{\infty})^m = BT^m.$$

Theorem 10.1. *There is a homotopy equivalence*

$$ET^m \times_{T^m} Z(K; (\underline{D}^{2J}, \underline{S}^{2J-1})) \xrightarrow{\alpha} Z(K; (\underline{\mathbb{C}P}^{\infty}, \underline{\mathbb{C}P}^{J-1})),$$

making the diagram following commute:

$$\begin{tikzcd}
ET^m \times_{T^m} Z(K; (\underline{D}^{2J}, \underline{S}^{2J-1})) \arrow[r, "\alpha"] \arrow[d, "p_1"] & Z(K; (\underline{\mathbb{C}P}^{\infty}, \underline{\mathbb{C}P}^{J-1})). \arrow[d, "i"'] \\
BT^m \arrow[r, "="] & BT^m.
\end{tikzcd}$$

Proof. For the pair $(\underline{X}, \underline{A}) = (\underline{D}^{2J}, \underline{S}^{2J-1})$, consider $D(\sigma)$ as in (6.1). The action of T^m leaves $D(\sigma)$ invariant, so

$$ET^m \times_{T^m} Z(K; (\underline{D}^{2J}, \underline{S}^{2J-1})) = ET^m \times_{T^m} \big(\cup_{\sigma \in K} D(\sigma)\big) = \bigcup_{\sigma \in K} ET^m \times_{T^m} D(\sigma).$$

The torus T^m acts diagonally, so it suffices to observe that

$$ET^1 \times_{T^1} D^{2j_k} \simeq \mathbb{C}P^{\infty} \quad \text{and} \quad ET^1 \times_{T^1} S^{2j_k-1} \simeq \mathbb{C}P^{j_k-1},$$

which follows because D^{2j_k} is contractible and T^1 acts freely on S^{2j_k-1}. So, the Borel construction converts $D(\sigma)$ for the family of pairs $(\underline{D}^{2J}, \underline{S}^{2J-1})$ into $D(\sigma)$ for the family of pairs $(\underline{\mathbb{C}P}^{\infty}, \underline{\mathbb{C}P}^{J-1})$. Moreover, the map α is constructed as a factorization of p_1, so the diagram does commute. $\square$

Remark 10.2. It is necessary to record certain standard facts about cell decompositions and their implications for the polyhedral product complexes $Z(K; (\underline{\mathbb{C}P}^{\infty}, \underline{\mathbb{C}P}^{J-1}))$. The classical example of C. Dowker [9], shows that some care is needed.

Let $J = (j_1, j_2, \ldots, j_m)$ be as above and fix $N > j_i - 1,\ i = 1, 2, \ldots, m$. For $\sigma \in K$, consider the space

$$D^N(\sigma) = \prod_{i=1}^{m} W_i, \quad \text{where} \quad W_i = \begin{cases} \mathbb{C}P^N & \text{if} \quad i \in \sigma \\ \mathbb{C}P^{j_i-1} & \text{if} \quad i \in [m] - \sigma. \end{cases} \tag{10.1}$$

The compact spaces $\mathbb{C}P^N$ and $\mathbb{C}P^{j_i-1}$, $i = 1, 2, \ldots, m$ are each assumed to be given the CW decomposition with one cell in each even dimension up to the top. This induces a cell decomposition of the product $D^N(\sigma)$, with each cell homeomorphic to a product of cells of even dimension, each in one of the spaces W_i. The compactness implies that the product topology and compactly generated topology agree.

Consider now the spaces $D(\sigma)$, (6.1), in $Z(K; (\underline{\mathbb{C}P}^\infty, \underline{\mathbb{C}P}^{J-1}))$:

$$D(\sigma) = \prod_{i=1}^{m} W_i, \quad \text{where} \quad W_i = \begin{cases} \mathbb{C}P^\infty & \text{if} \quad i \in \sigma \\ \mathbb{C}P^{j_i-1} & \text{if} \quad i \in [m] - \sigma. \end{cases} \tag{10.2}$$

The spaces $D(\sigma)$ are each a colimit, over increasing N, of the spaces $D^N(\sigma)$. The colimit is via compatible inclusions and so each space $D(\sigma)$ inherits a CW structure with cells in even dimension. Finally,

$$Z(K; (\underline{\mathbb{C}P}^\infty, \underline{\mathbb{C}P}^{J-1})) \;=\; \bigcup_{\sigma \in K} D(\sigma)$$

is a finite colimit of spaces which have compatible cell structures on intersections, so inherits a cell structure with cells concentrated in even dimension.

From these considerations follows the next lemma.

Lemma 10.3. *The inclusion of the subcomplex $Z(K; (\underline{\mathbb{C}P}^\infty, \underline{\mathbb{C}P}^{J-1})) \subseteq BT^m$ induces an surjective map*

$$H^*(BT^m; \mathbb{Z}) \longrightarrow H^*\big(Z(K; (\underline{\mathbb{C}P}^\infty, \underline{\mathbb{C}P}^{J-1})); \mathbb{Z}\big).$$

Let I_K^J be the ideal in $\mathbb{Z}[v_1, v_2, \ldots, v_m]$ described in Theorem 4.2. It is generated by all monomials $v_{i_1}^{j_{i_1}} v_{i_2}^{j_{i_2}} \cdots v_{i_k}^{j_{i_k}}$ corresponding to minimal non-simplices $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ of K.

Definition 10.4. Let K be a simplicial complex on m vertices and $J = (j_1, j_2, \ldots, j_m)$, a sequence of positive integers. The ring $\mathbb{Z}[v_1, v_2, \ldots, v_m]/I_K^J$ is called the J-weighted Stanley-Reisner ring of K and is denoted by the symbol $SR^J(K)$. The space $Z(K; (\underline{\mathbb{C}P}^\infty, \underline{\mathbb{C}P}^{J-1}))$ is called the J-weighted Davis-Januszkiewicz space and is denoted by $\mathcal{DJ}^J(K)$.

Theorem 10.5. *There is an isomorphism of rings*

$$H^*(Z(K;(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{J-1}));\mathbb{Z}) \longrightarrow \mathbb{Z}[v_1,v_2,\dots,v_m]/I_K^J.$$

Proof. The proof is a modification of the inductive argument from [7], (Theorem 4.8 and Lemma 4.9), where the authors compute $H^*(\mathcal{DJ}(K);\mathbb{Z})$. Consider first $K=\Delta^{m-1}$ an $(m-1)$-simplex. The Borel fibration in this case is the disc bundle

$$D^{2d(J)} \;\cong\; \prod_{i=1}^{m} D^{2j_i} \longrightarrow ET^m \times_{T^m} Z(\Delta^{m-1};(\underline{D}^{2J},\underline{S}^{2J-1})) \longrightarrow BT^m \tag{10.3}$$

and the associated sphere bundle is

$$S^{2d(J)-1} \;\cong\; \partial(\prod_{i=1}^{m} D^{2j_i}) \longrightarrow ET^m \times_{T^m} Z(\partial(\Delta^{m-1});(\underline{D}^{2J},\underline{S}^{2J-1})) \longrightarrow BT^m. \tag{10.4}$$

Since T^m acts diagonally, (7.3), the vector bundle associated to (10.3) is a sum of complex line bundles $j_1L_1\oplus j_2L_2\oplus\cdots\oplus j_mL_m$, where L_i is the line bundle having first Chern class given by $c_1(L_i)=v_i\in H^2(BT^m;\mathbb{Z})$. The Euler class of this bundle is given by the Whitney formula

$$c_{d(J)}(j_1L_1\oplus j_2L_2\oplus\cdots\oplus j_mL_m) \;=\; v_1^{j_1}v_2^{j_2}\cdots v_m^{j_m} \;\in\; H^{2d(J)}(BT^m;\mathbb{Z}). \tag{10.5}$$

The associated Gysin sequence is

$$\begin{aligned}\cdots\to H^{k+2d(J)-1}\big(ET^m\times_{T^m} Z(\partial(\Delta^{m-1});(\underline{D}^{2J},\underline{S}^{2J-1}))\big) &\longrightarrow H^k(BT^m) \xrightarrow{\cdot c_{d(J)}} \\ H^{k+2d(J)}(BT^m) \longrightarrow H^{k+2d(J)}\big(ET^m\times_{T^m} Z(\partial(\Delta^{m-1});&(\underline{D}^{2J},\underline{S}^{2J-1}))\big)\to\cdots\end{aligned} \tag{10.6}$$

The sequence implies that for $K=\partial(\Delta^{m-1})$, the kernel of the surjection

$$H^*(BT^m;\mathbb{Z}) \longrightarrow H^*\big(Z(K;(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{J-1}));\mathbb{Z}\big),$$

is exactly the ideal in $\mathbb{Z}[v_1,v_2,\dots,v_m]$ generated by all monomials $v_{i_1}^{j_{i_1}}v_{i_2}^{j_{i_2}}\cdots v_{i_m}^{j_{i_m}}$, showing that Theorem 10.5 holds for K equal to the boundary of a simplex.

Suppose next that $K^0=\big\{\{v_1\},\{v_2\},\dots,\{v_m\}\big\}$ is a discrete set of m points. In this case

$$Z(K^0; (\underline{\mathbb{C}P}^{\infty}, \underline{\mathbb{C}P}^{J-1})) \;\cong\; \bigcup_{i=1}^{m} W_i \;\subset\; (\mathbb{C}P^{\infty})^m$$

where

$$W_i \;=\; \mathbb{C}P^{j_1-1} \times \cdots \times \mathbb{C}P^{j_{i-1}-1} \times \mathbb{C}P^{\infty} \times \mathbb{C}P^{j_{i+1}-1} \times \cdots \times \mathbb{C}P^{j_m-1}.$$

Notice that the composition $W_i \longrightarrow Z(K^0; (\underline{\mathbb{C}P}^{\infty}, \underline{\mathbb{C}P}^{J-1})) \longrightarrow (\mathbb{C}P^{\infty})^m$, factors the natural inclusion $W_i \longrightarrow (\mathbb{C}P^{\infty})^m$. This implies that all monomials of the form

$$v_1^{j_1-1} \cdots v_{i-1}^{j_{i-1}-1} v_i^k v_{i+1}^{j_{i+1}-1} \cdots v_m^{j_m-1}$$

are non-zero in $H^*\big(Z(K^0; (\underline{\mathbb{C}P}^{\infty}, \underline{\mathbb{C}P}^{J-1}))\big)$ for all i and k. On the other hand, the inclusion $\big\{\{v_i\}, \{v_k\}\big\} \hookrightarrow K^0$, is an inclusion of a full sub-complex for each pair $\{i, k\}$. The conclusion of [8, Lemma 2.2.3] implies now that for $\hat{J} = (j_i, j_k)$, the induced map

$$Z\big(\big\{\{v_i\}, \{v_k\}\big\}; (\underline{\mathbb{C}P}^{\infty}, \underline{\mathbb{C}P}^{\hat{J}-1})\big) \longrightarrow Z(K^0; (\underline{\mathbb{C}P}^{\infty}, \underline{\mathbb{C}P}^{J-1}))$$

is a retraction. Since $\big\{\{v_i\}, \{v_k\}\big\}$ is the boundary of a simplex, the previous result implies that $v_i^{j_i} v_k^{j_k} = 0$ in $H^*(Z(K^0; (\underline{\mathbb{C}P}^{\infty}, \underline{\mathbb{C}P}^{J-1})); \mathbb{Z})$. So, Theorem 10.5 holds for K equal to a discrete set of m points.

Consider now any simplicial complex K of dimension $n-1$ on m vertices $\{v_1, v_2, \ldots, v_m\}$ and suppose that the conclusion of Theorem 10.5 holds for the $(n-2)$-skeleton K^{n-2}. The complex $K = K^{n-1}$ will be constructed by attaching one $(n-1)$-simplex Δ^{n-1} at a time to K^{n-2}. Let $\{v_{i_1}, v_{i_2}, \ldots, v_{i_n}\}$ be the vertices of the full-subcomplex $\partial(\Delta^{n-1}) \subset K^{n-2}$ and $\{v_{k_1}, v_{k_2}, \ldots, v_{k_{m-n}}\}$ the complementary set of vertices in $\{v_1, v_2, \ldots, v_m\}$. Corresponding to these sets, define sequences $J' = (j_{i_1}, j_{i_2}, \ldots, j_{i_n})$ and $J'' = (j_{k_1}, j_{k_2}, \ldots, j_{k_{m-n}})$, so that J is the amalgamation of J' and J''.

Again, [8, Lemma 2.2.3] implies that

$$Z(\partial(\Delta^{n-1}); (\underline{\mathbb{C}P}^{\infty}, \underline{\mathbb{C}P}^{J'-1})) \longrightarrow Z(K^{n-2}; (\underline{\mathbb{C}P}^{\infty}, \underline{\mathbb{C}P}^{J-1}))$$

is a retraction. It follows that there is an isomorphism of groups

$$H^*\big(Z(K^{n-2};(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{J-1}))\big) \;\cong\; A^* \oplus H^*\big(Z(\partial(\Delta^{n-1});(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{J'-1}))\big), \tag{10.7}$$

for some graded group A^*. Recall next, that from the colimit description,

$$Z\big(K^{n-2}\cup\Delta^{n-1};(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{J-1})\big) \;=\; Z\big(K^{n-2};(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{J-1})\big) \,\cup\, D(\Delta^{n-1}) \tag{10.8}$$

where, as in (6.1),

$$D(\Delta^{n-1}) = \prod_{i=1}^{m} Y_i, \quad \text{where} \quad Y_i = \begin{cases} \mathbb{C}P^\infty & \text{if } \; i\in\Delta^{n-1} \\ \mathbb{C}P^{j_i-1} & \text{if } \; i\in[m]-\Delta^{n-1}. \end{cases}$$

Notice that in the notation J' and J'' above, up to a rearrangement of factors

$$D(\Delta^{n-1}) = B\times\prod_{s=i_1}^{i_n}\mathbb{C}P^\infty.$$

where $B = \mathbb{C}P^{j_{k_1}}\times\mathbb{C}P^{j_{k_2}}\times\cdots\times\mathbb{C}P^{j_{k_{m-n}}}$. Now since $\partial(\Delta^{n-1})$ is a full sub-complex of K^{n-2}, the union in (10.8) is along the subspace

$$Z(\partial(\Delta^{n-1});(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{J'-1}))\times B \tag{10.9}$$

(cf. [12, Corollary 3.2]). On making the identification (10.7) and noting that everything is concentrated in even degree, the Mayer-Vietoris sequence associated to (10.8) becomes

$$\begin{aligned} 0 \;\longrightarrow\; & H^{2k}\big(Z\big(K^{n-2}\cup\Delta^{n-1};(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{J-1})\big)\big) \xrightarrow{\;\psi\;} \\ & \big[A^{2k} \;\oplus\; H^{2k}\big(Z(\partial(\Delta^{n-1});(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{J'-1}))\big)\big] \;\oplus\; H^{2k}\big(B\times\prod_{s=i_1}^{i_n}\mathbb{C}P^\infty\big) \xrightarrow{\;\phi\;} \\ & H^{2k}\big(Z(\partial(\Delta^{n-1});(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{J'-1}))\times B\big) \longrightarrow 0 \end{aligned}$$

where ψ is induced by restriction and ϕ by the difference in restrictions to the intersection. The space $\prod_{s=i_1}^{i_n}\mathbb{C}P^\infty$ is not in the intersection (10.9) and so the class

$$v_{i_1}^{j_{i_1}}v_{i_2}^{j_{i_2}}\cdots v_{i_n}^{j_{i_n}} \;\in\; H^*\big(\prod_{s=i_1}^{i_n}\mathbb{C}P^\infty\big)$$

maps to zero under ϕ. Finally, regarding $H^*\big(Z\big(K^{n-2}\cup\Delta^{n-1};(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{J-1})\big)\big)$ as a quotient of $H^*(BT^m)$, as in Lemma 10.3, it follows that, the relation $v_{i_1}^{j_{i_1}}v_{i_2}^{j_{i_2}}\cdots v_{i_n}^{j_{i_n}} = 0$ given by the induction hypothesis, is removed from $H^*(Z(K^{n-2};(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{J-1}));\mathbb{Z})$, and results in a description of the cohomology ring $H^*\big(Z\big(K^{n-2}\cup\Delta^{n-1};(\underline{\mathbb{C}P}^\infty,\underline{\mathbb{C}P}^{J-1})\big)\big)$ consistent with Theorem 10.5. This completes the inductive step and the proof. $\square$

11. Generalized moment-angle complexes and the cohomology of $M(J)$

The computation of $H^*(M^{2n};\mathbb{Z})$ in [7] is generalized to recover Theorem 4.2 directly from the results of the previous section. Traditionally, ([7], [6]), the canonical diagram of fibrations

$$\begin{array}{ccccc}
\ker\lambda(J) & \longrightarrow & E(\ker\lambda(J)) & \longrightarrow & B(\ker\lambda(J)) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
Z(K(J);(D^2,S^1)) & \longrightarrow & ET^{d(J)}\times_{T^{d(J)}} Z(K(J);(D^2,S^1)) & \xrightarrow{p_1'} & BT^{d(J)} \\
\big\downarrow{\scriptstyle r'} & & \big\downarrow{\scriptstyle q'\,\simeq} & & \big\downarrow{\scriptstyle B(\lambda(J))} \\
M(J) & \xrightarrow{i'} & ET^{d(J)-m+n}\times_{T^{d(J)-m+n}} M(J)) & \xrightarrow{p_2'} & BT^{d(J)-m+n}
\end{array}$$

is used, in conjunction with the Serre spectral sequence of the fibration in the bottom row, to obtain the standard description of $H^*(M^{2n};\mathbb{Z})$ given by Theorem 4.1.

The more condensed calculation in Theorem 4.2 is obtained by considering instead the homotopy commutative diagram of fibrations

$$\begin{array}{ccccc}
\ker\lambda(J) & \longrightarrow & E(\ker\lambda(J)) & \longrightarrow & B(\ker\lambda(J)) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
Z(K;(\underline{D}^{2J},\underline{S}^{2J-1})) & \longrightarrow & ET^m\times_{T^m} Z(K;(\underline{D}^{2J},\underline{S}^{2J-1})) & \xrightarrow{p_1} & BT^m \\
\big\downarrow{\scriptstyle r} & & \big\downarrow{\scriptstyle q\,\simeq} & & \big\downarrow{\scriptstyle s} \\
M(J) & \xrightarrow{i} & ET^n\times_{T^n} M(J)) & \xrightarrow{p_2} & BT^n.
\end{array} \tag{11.1}$$

In the diagram, r is the map given by Theorem 9.2 and the map s is determined by (9.4). The equivalence q is a consequence of the splitting, $T^m \cong \ker\lambda(J)\times T^n$ as topological groups and the fact that $\ker\lambda(J)$ acts freely on $Z(K;(\underline{D}^{2J},\underline{S}^{2J-1}))$.

Theorem 10.1 allows the replacement, up to homotopy, of the fibration along the bottom row of the diagram with

$$M(J)\longrightarrow Z(K;(\underline{\mathbb{C}P}^{\infty},\underline{\mathbb{C}P}^{J-1}))\xrightarrow{p} BT^n. \tag{11.2}$$

These observations allow for an alternative approach to the calculation of Theorem 4.2.

Theorem 11.1. *There is an isomorphism of rings*

$$H^*(M(J);\mathbb{Z}) \longrightarrow H^*\big(Z(K;(\underline{\mathbb{C}P^\infty},\underline{\mathbb{C}P^{J-1}}));\mathbb{Z}\big)/L,$$

where L is the two-sided ideal generated by the image of the map p^*.

Proof. In the Serre spectral sequence associated with (11.2), all groups are concentrated in even degree. This is true for $H^*(M(J);\mathbb{Z})$ because $M(J)$ is a toric manifold by Theorem 3.1 and, for $H^*\big(Z(K;(\underline{\mathbb{C}P^\infty},\underline{\mathbb{C}P^{J-1}}));\mathbb{Z}\big)$, by Theorem 10.5. The spectral sequence collapses. The E_2-term is given by

$$H^*(M(J)) \otimes H^*(BT^n).$$

It follows that $H^*(M(J))$ is the quotient of $H^*\big(Z(K;(\underline{\mathbb{C}P^\infty},\underline{\mathbb{C}P^{J-1}}))\big)$ by the two-sided ideal generated by the image of p^*. $\square$

It remains to identify the ideal L in Theorem 11.1. With reference to the right-hand bottom square in diagram (11.1), the map p^* is the composition $(\alpha^{-1})^* \circ (p_1^* \circ s^*)$, where the map α is the equivalence of Theorem 10.1. So, the image of p^* is the image of the composition

$$\begin{array}{ccccc}
H^*(BT^n) & \xrightarrow{s^*} & H^*(BT^m) & \longrightarrow & SR^J(K) \\
\big\downarrow \cong & & \big\downarrow \cong & & \big\downarrow \cong \\
\mathbb{Z}[u_1,u_2,\ldots,u_n] & \longrightarrow & \mathbb{Z}[v_1,v_2,\ldots,v_m] & \longrightarrow & \mathbb{Z}[v_1,v_2,\ldots,v_m]/I_K^J.
\end{array}$$

This is specified by (9.4) and generates the ideal L_M of Theorem 4.2, determined by the rows of the original matrix λ. So, $L = L_M$ and Theorem 11.1 becomes

$$H^*(M(J);\mathbb{Z}) \;\cong\; \mathbb{Z}[v_1, v_2, \ldots, v_m]/(I_K^J + L_M), \tag{11.3}$$

recovering Theorem 4.2 completely from the topological point of view of generalized moment-angle complexes.

12. Nests

Let $J = (j_1, j_2 \ldots, j_m)$ and $L = (l_1, l_2, \ldots, l_m)$ be sequences of positive integers. Define an ordering on such sequences by putting $J < L$ if $j_i \leq l_i$ for all i and, for at least one $k \in \{1, 2, \ldots, m\}$, $j_k < l_k$. If $J < L$ the inclusions $D^{2j_i} \subseteq D^{2l_i}$ induce an equivariant embedding

$$Z(K; (\underline{D}^{2J}, \underline{S}^{2J-1})) \;\subset\; Z(K; (\underline{D}^{2L}, \underline{S}^{2L-1}))$$

and, consequently, an embedding $\zeta\colon M(J) \longrightarrow M(L)$. The next proposition follows easily.

Proposition 12.1. *For $J < L$, the normal bundle of the embedding ζ is*

$$\bigoplus_{i=1}^{m} (l_i - j_i)\alpha_i, \tag{12.1}$$

where α_i is the line bundle over $M(J)$ with first Chern class $c_i(\alpha_i) = v_i$, the class from the cohomology description (11.3). *Moreover, the induced map*

$$i^*\colon H^k(M(L);\mathbb{Z}) \longrightarrow H^k(M(J);\mathbb{Z})$$

is an epimorphism which is an isomorphism for $k = 2$.

Proof. Diagram (11.1) implies that the diagram below commutes up to homotopy. (Notice here that the rows are not fibrations up to homotopy.)

$$\begin{array}{ccccc}
M(J) & \xrightarrow{\;i\;} & ET^m \times_{T^m} Z(K; (\underline{D}^{2J}, \underline{S}^{2J-1})) & \xrightarrow{\;p_1^J\;} & BT^m \\
\downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \bar{\zeta}} & & \downarrow \\
M(L) & \xrightarrow{\;i\;} & ET^m \times_{T^m} Z(K; (\underline{D}^{2L}, \underline{S}^{2L-1})) & \xrightarrow{\;p_1^L\;} & BT^m.
\end{array}$$

Theorems 10.1 and 10.5 imply that $\overline{\zeta}^*$ is onto and the Serre spectral sequence part of the proof of Theorem 11.1 shows that i^* is onto. This implies that ζ^* is onto too. The first part of the proposition follows from the fact that each canonical bundle L_i over BT^m pulls back to the bundle $l_i\alpha_i$ over $M(L)$ and to $j_i\alpha_i$ over $M(J)$. □

A nest of toric manifolds is constructed from a toric manifold triple (P^n, M^{2n}, λ), an initial sequence $J_0 = (1, 1, \dots, 1)$ with m entries corresponding to the number of facets of P^{2n} and a sequence $J_0 < J_1 < J_2 < \cdots$ with the property that $d(J_{i+1}) = d(J_i) + 1$. (The symbol $d(J)$ is defined at the end of Section 2).

Proposition 12.2. *Given a toric manifold (P^n, M^{2n}, λ) and a sequence $J_0 < J_1 < J_2 < \cdots$ as above, there is a nest of toric manifolds,*

$$M^{2n} = M(J_0) \subset M(J_1) \subset \cdots \subset M(J_k) \subset \dots$$

where $d(J_i) = m + i$, the dimension of $M(J_i)$ is $2n + 2i$ and $M(J_i) \subset M(J_{i+1})$ is a codimension-two embedding. Furthermore, there is a sequence of epimorphisms

$$\cdots \longrightarrow H^k(M(J_i); \mathbb{Z}) \longrightarrow H^k(M(J_{i-1}); \mathbb{Z}) \longrightarrow \cdots \longrightarrow H^k(M^{2n}; \mathbb{Z})$$

which are isomorphisms for $k = 2$. □

Problem: Beginning with a toric manifold (P^n, M^{2n}, λ), find invariants, possibly in terms of λ, which will detect diffeomorphic, (homotopic), manifolds in nests corresponding to different sequences $J_0 < J_1 < J_2 < \cdots$.

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